AD-A044 981 STANFORD UNIV CALIF DEPT OF OPERATIONS RESEARCH F/G 12/2

THE GI/M/1 QUEUE WITH CONTROLLED ARRIVALS.(U)

JUL 77 S STIDHAM N00014-76-C-0418

UNCLASSIFIED TR-74 NL

1 OF 1

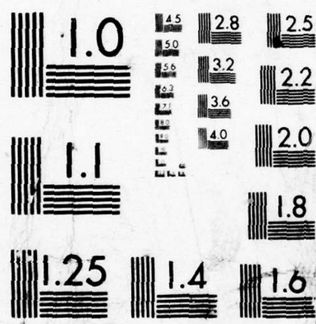
AD A044981

END

DATE FILMED

11-77

DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

12

THE GI/M/1 QUEUE WITH CONTROLLED ARRIVALS

BY

SHALER STIDHAM, JR.

TECHNICAL REPORT NO. 74
JULY 21, 1977

PREPARED UNDER CONTRACT
N00014-76-C-0418 (NR-047-061)
FOR THE OFFICE OF NAVAL RESEARCH

Frederick S. Hillier, Project Director

This research was supported in part by National Science Foundation Grant ENG 75-14847 Department of Operations Research, Stanford University and issued as Technical Report #44. Also, supported in part by a grant from the Bank of America Foundation at Stanford University.

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

(See 1473)

Abstract

A GI/M/1 queue is studied, in which the arrival process can be controlled by accepting or rejecting arriving customers or changing a toll. An entering customer receives a (random) reward and there is a holding cost, which is convex in the number of customers present. We compare socially and individually optimal joining policies for finite and infinite horizons, with and without discounting. We show that a socially optimal policy is less likely to accept a customer than an individually optimal policy, and both policies are less likely to accept as the number of customers present increases, the horizon length increases, or the discount rate decreases. The properties of optimal congestion tolls are examined and it is discovered that the toll is not monotonic when either the horizon is finite or the discount rate is positive, whereas it can be monotonic for infinite-horizon, average-return problems.

Introduction

The subject of this paper is GI/M/1 queue in which the arrival process can be controlled by accepting or rejecting arriving customers. There is a random reward earned when a customer enters the system and a convex holding cost for the customers already present. We study the properties of individually and socially optimal joining policies for finite- and infinite-horizon problems, both with and without discounting. The properties of optimal congestion tolls are also examined.

Naor [10] was the first to compare individually and socially optimal policies and to demonstrate that, left to their own devices (that is, under an individually optimal policy), customers will enter the system more often than is optimal for the collective of all customers (that is, under a socially optimal policy). Naor considered an M/M/1 system operating under a critical-number policy, with deterministic reward and linear holding cost and expected steady-state net benefit as the criterion. Naor's results were extended to more general systems in references [1], [4], [5], [9], [13], and [14]. The most relevant references to the present paper are Yechiali [13] and Lippman and Stidham [9]. The former considered GI/M/1 systems, but only in steady-state and only with deterministic rewards and linear holding costs. The latter considered exponential congestion systems (including the M/M/1 system as a special case), with random rewards, finite- and infinite-horizons, and discounted return and average return as criteria. The present paper may be regarded as an extension of both these references.

In Section 1 we formulate as a semi-Markov decision process the problem of finding a socially optimal policy over a finite-horizon. The socially optimal policy is shown to be monotonic, in the sense that an arriving customer is less likely to enter as the number of customers already present increases, the length of the horizon increases, or the interest rate decreases. The model differs from previous ones (e.g., reference [9]) in that it allows a

non-zero terminal reward, or cost. This feature makes it possible, for example, to model the situation in which the server continues to operate until all customers present at the end of the horizon have been served.

In Section 2 socially optimal and individually optimal joining policies are compared. The difference between the joining parameters for the two policies is shown to equal the external effect of a customer's decision to enter on the costs incurred by future customers. As expected, this external effect is positive, so that an individually optimal policy allows customers to enter who should not enter, if they took into account the costs imposed thereby on other customers.

Optimal congestion tolls - entrance fees that induce individual customers to act in accordance with a socially optimal policy - are studied in Section 3. It is shown that, contrary to intuition, the optimal toll is asymptotically zero and hence not monotonic in the number of customers in the system. The same result was discovered by Lippman and Stidham [9] in the context of exponential systems, in which the system is observed at epochs of service completion and "null events" (see Lippman [7]) as well as arrivals. The reason for the phenomenon, however, is somewhat different and more basic in the present context. Our analysis sheds light on why one should never expect monotonic tolls in finite-horizon problems, even when there is no discounting and the server continues to operate after the end of the horizon.

Infinite-horizon problems are discussed in Section 4. The monotonic properties of a socially optimal policy are shown to carry over to the infinite-horizon problem with discounting, with no additional restrictions on the reward distribution or holding-cost function. Under a slight restriction on the holding-cost function (weaker than the restrictions in Lippman [8]) the method of successive approximations is shown to converge, when there is a terminal cost equal to the expected discounted holding cost incurred until all customers in the system at the end of the horizon are served. For the

infinite-horizon, average-return problem with linear holding-cost, monotonicity of the socially optimal policy is demonstrated using the methods of Lippman [8].

Finally, we study optimal congestion tolls for infinite-horizon problems, both with and without discounting. Functional equations and suggestions for computing optimal tolls are given for both cases. In the discounted case, we show once again that the toll is asymptotically zero and hence cannot be monotonic. By contrast, in the average-return case, we show that the toll is asymptotically positive and, at least in the M/M/1 case, monotonically increasing in the number of customers present. This discrepancy between optimal policies with and without discounting implies, among other things, that there is no strongly (or Blackwell) optimal policy.

1. Socially Optimal Policies

We consider a GI/M/1 queueing system that can be controlled by accepting or rejecting arriving customers. Specifically, the times between the arrivals of successive customers are independent and identically distributed as a random variable T, with $0 < E\{T\} < \infty$. The successive service times of customers who enter are independent and exponentially distributed with parameter $\mu > 0$. Associated with each customer is a random reward, or utility of service, which is received if he enters the system. The rewards of successive customers are independent and identically distributed as a random variable R with $E\{|R|\} < \infty$. There is also a deterministic holding cost, incurred at rate $h(k)$ while there are k customers in the system, where h is a convex, non-decreasing function. Costs and rewards are continuously discounted at rate α ($\alpha \geq 0$), so that the present value of x dollars earned at time t is exp $(-\alpha t)$.

The objective of a social optimizer is to maximize the expected α-discounted net benefit (rewards minus holding costs) to society as a whole, over a finite or infinite horizon. We formulate the problem as a semi-Markov decision process [11], [6], [8], in which the decision points are the times at which customers arrive and the state of the system at an arrival point is denoted (i,r), where i = number of customers present (not including the customer seeking admittance) and r = reward of the customer seeking admittance. The possible actions are $a = 1$ (admit the customer) and $a = 0$ (reject the customer).

Let $V_{n,\alpha}(i,r)$ $(n \geq 0)$ be the maximal expected α-discounted net benefit to society as a whole when a customer with reward r has just arrived to a system with i customers already present, and the horizon length is n periods. That is to say, the system will operate until n additional customers have arrived and then admit no further customers. We assume that at the end of the

horizon (that is, after the arrival of the n^{th} customer), the system earns a terminal benefit $U_{o,\alpha}(i)$, depending on the number of customers, i, still in the system. For example, the server might continue to operate until all i customers have been served, in which case $U_{o,\alpha}(i)$ would be minus the expected α-discounted holding cost incurred until no customers remain to be served.

The $V_{n,\alpha}(i,r)$ satisfy the following recursive equations:

(1) $$V_{n,\alpha}(i,r) = \max\{r + U_{n,\alpha}(i+1);\ U_{n,\alpha}(i)\},$$

for $n \geq 0$, where

(2) $$U_{n,\alpha}(i) = -E\{\int_0^T e^{-\alpha t} h((i - N(t))^+)dt\} + E\{e^{-\alpha T} V_{n-1,\alpha}((i - N(T))^+, R)\},$$

$n \geq 1$, and $\{N(t), t \geq o\}$ is a Poisson process with mean rate μ, independent of T. N(t) is the number of potential service completions in [0,t], that is, the number of services which could be completed by time t if the server continued to perform "services" when no customers were present. The first term on the right-hand side of (2) is minus the expected α-discounted holding cost incurred until the next arrival (decision) point. The second term is the expected present value of the net benefit from following an optimal admission policy from the next decision point until the end of the horizon.

Let $S_{n,\alpha}(i) \equiv U_{n,\alpha} - U_{n,\alpha}(i+1)$, $n \geq 0$, $i \geq 0$. It follows immediately from (1) that the socially optimal policy takes the *form: admit a customer* when i customers are already present and n decision points remain if and only if

(3) $$r \geq S_{n,\alpha}(i).$$

The remainder of this section is concerned with verifying the anticipated monotonicity of socially optimal policies. Specifically, we show that a socially optimal policy is less likely to admit an arriving customer as i, the number of customers already present, increases (Theorem 3), as n, the number

of remaining decision points, increases (Theorem 5), or as α, the discount rate, decreases (Theorem 6).

For our first result (Theorem 3) we need the following two lemmas. The first is from Lippman and Stidham [9] and the second expresses well-known and easily verified properties of convex and concave functions.

Lemma 1. Let $g(i) \equiv \max\{r + f(i+1);\ f(i)\}$, $i = 0, 1, \ldots$. If $f(i)$ is concave and non-increasing in i, then $g(i)$ is concave and non-increasing in i.

Proof. See Lemma 2 of [9].

Lemma 2. Let $\phi(i) = g(f(i))$, $i = 0, 1, \ldots$, where $f(i)$ is a convex, non-decreasing, integer-valued function of $i = 0, 1, \ldots$, with $f(0) \geq 0$, and $g(j)$ is a concave, non-increasing function of $j = 0, 1, \ldots$. Then $\phi(i)$ is a concave function non-increasing of $i = 0, 1, \ldots$.

We are now in a position to prove:

Theorem 3. Suppose $U_{o,\alpha}(i)$ is concave and non-increasing in i. Then, for each r, $\alpha > 0$, $n \geqq 0$, $V_{n,\alpha}(i,r)$ is concave and non-increasing in i. Moreover, $0 \leqq S_{n,\alpha}(i) \leqq S_{n,\alpha}(i+1)$, $i \geqq 0$.

Proof. The proof is by induction on n. The case $n = 0$ follows immediately from (1), the hypothesis, and Lemma 1. Now suppose $n \geqq 1$ and $V_{n-1,\alpha}(i,r)$ is concave and non-increasing in i.

Since $h(i)$, $(i - N(t))^+$, and $(i - N(T))^+$ are convex and non-decreasing in i (for each fixed value of t, $N(t)$, T, and $N(T)$), it follows from (2) and Lemma 2 that $U_{n,\alpha}(i)$ is concave and non-increasing in i. As a consequence, $S_{n,\alpha}(i)$ is non-negative and non-decreasing in i. Finally, Lemma 1 implies that $V_{n,\alpha}(i,r)$ is concave and non-increasing in i, which completes the proof.

For the remainder of this section we restrict attention to the special case in which the terminal reward, $U_{o,\alpha}(i)$, is minus the expected α-discounted holding cost incurred until the i customers remaining in the system are served.

That is,

$$U_{o,\alpha}(i) = -E\{\int_0^\infty e^{-\alpha t} h((i - N(t))^+)dt\}. \tag{4}$$

Since h(.) is convex and non-decreasing, $U_{o,\alpha}(i)$ is concave and non-increasing in this case. The following corollary is then an immediate consequence of Theorem 3.

Corollary 4. Suppose $U_{o,\alpha}(i)$ is given by (4). Then $V_{n,\alpha}(i,r)$ is concave and non-increasing in i and $0 \leq S_{n,\alpha}(i) \leq S_{n,\alpha}(i,1)$, $i \geq 0$.

Define $v_{n,\alpha}(i,r) \equiv V_{n,\alpha}(i,r) - V_{n,\alpha}(i+1,r)$. It follows from (1) and Theorem 3 that $(n \geq 0)$

$$v_{n,\alpha}(i,r) = \begin{cases} S_{n,\alpha}(i), & r < S_{n,\alpha}(i) \\ r & , S_{n,\alpha}(i) \leq r < S_{n,\alpha}(i+1) \\ S_{n,\alpha}(i+1), & r \geq S_{n,\alpha}(i,1). \end{cases} \tag{5}$$

From (2) it follows that $(n \geq 1)$

$$\begin{aligned} S_{n,\alpha}(i) &= -E\{\int_0^T e^{-\alpha t} [h((i - N(t))^+) - h((i + 1 - N(t))^+)] dt\} \\ &\quad +E\{e^{-\alpha T} [V_{n-1,\alpha}((i - N(T))^+, R) - V_{n-1,\alpha}((i + 1 - N(T))^+, R)]\} \\ &= -E\{\int_0^T e^{-\alpha t}[h((i - N(t))^+) - h((i + 1 - N(t))^+)] dt\} \\ &\quad +E\{e^{-\alpha T} v_{n-1,\alpha}(i - N(T), R); N(T) \leq i\}. \end{aligned} \tag{6}$$

Theorem 5. Suppose $U_{o,\alpha}(i)$ is given by (4). Given $\alpha \geq 0$, i, and r, the functions $v_{n,\alpha}(i,r)$ and $S_{n,\alpha}(i)$ are non-decreasing in $n \geq 0$.

Proof. The proof is by induction on n. For n = 0, (5) implies that $v_{o,\alpha}(i,r) \geq S_{o,\alpha}(i) = U_{o,\alpha}(i) - U_{o,\alpha}(i + 1)$. Combining this inequality with (6) for n = 1, we conclude that

(7) $S_{1,\alpha}(i) \geq -E\{ \int_0^T e^{-\alpha t}[h((i - N(t))^+) - h((i + 1 - N(t))^+] dt$

$$+E\{e^{-\alpha T}[U_{o,\alpha}(i - N(T)) - U_{o,\alpha}(i + 1 - N(T))]; N(T) \leq i\}$$

From (4) it follows that

(8) $U_{o,\alpha}(i) = -E\{ \int_0^T e^{-\alpha t} h((i - N(t))^+)dt\} - E\{ \int_T^\infty e^{-\alpha t}h((i - N(t))^+)dt\}.$

Let $\{N'(t), t \geq 0\}$ be a Poisson process with mean rate μ, independent of T and $\{N(t), t \geq 0\}$. Then

(9) $E\{ \int_T^\infty e^{-\alpha t} h((i - N(t))^+) dt\}$

$$= E\{e^{-\alpha T} \int_T^\infty e^{-\alpha(t-T)}h((i - N(T) - (N(t) - N(T)))^+)dt\}$$

$$= E\{e^{-\alpha T} \int_T^\infty e^{-\alpha(t-T)}h((i - N(T) - N'(t-T))^+)dt\}$$

$$= E\{e^{-\alpha T} \int_0^\infty e^{-\alpha t} h((i - N(T) - N'(t))^+)dt\}$$

$$= E\{E\{e^{-\alpha T} \int_0^\infty e^{-\alpha t} h((i - N(T) - N'(t))^+)dt|T,N(T)\}\}$$

$$= E\{e^{-\alpha T} E\{ \int_0^\infty e^{-\alpha t} h((i-N(T) - N'(t))^+)dt\}\}$$

$$= -E\{e^{-\alpha T} U_{o,\alpha}((i - N(T))^+)\}.$$

Combining (8) and (9), we conclude that

(10) $U_{o,\alpha}(i) = -E\{ \int_0^T e^{-\alpha t} h((i - N(t))^+) dt\} + E\{e^{-\alpha T} U_{o,\alpha}((i - N(T))^+)\}.$

Therefore,

(11) $S_{o,\alpha}(i) = U_{o,\alpha}(i) - U_{o,\alpha}(i + 1)$

$$= -E\{ \int_0^T e^{-\alpha t}[h((i - N(t))^+) - h((i + 1 - N(t))^+)]dt$$

$$+E\{e^{-\alpha T}[U_{o,\alpha}((i - N(T))^+) - U_{o,\alpha}((i + 1 - N(T))^+)]\}$$

$$= -E\{\int_0^T e^{-\alpha t}[h((i - N(t))^+) - h((i + 1 - N(t))^+)]dt$$

$$+E\{e^{-\alpha T}[U_{o,\alpha}(i - N(T)) - U_{o,\alpha}(i + 1 - N(T))];\ N(T) \leq i\}$$

Together, (7) and (11) imply that $S_{1,\alpha}(i) \geq S_{o,\alpha}(i)$. From (5) we then conclude that $v_{1,\alpha}(i,r) \geq v_{o,\alpha}(i,r)$.

Turning now to the general case of $n \geq 1$, suppose $v_{n,\alpha}(i,r) \geq v_{n-1,\alpha}(i,r)$. Then from (6) it follows that $S_{n+1,\alpha}(i) \geq S_{n,\alpha}(i)$, which in turn implies (from (5)) that $v_{n+1,\alpha}(i,r) \geq v_{n,\alpha}(i,r)$. This completes the proof of the theorem.

<u>Theorem 6.</u> Suppose $U_{o,\alpha}(i)$ is given by (4). Given n, α, r, the functions $v_{n,\alpha}(i,r)$ and $S_{n,\alpha}(i)$ are strictly decreasing in $\alpha > 0$.

<u>Proof.</u> The proof is by induction on n, using (4), (5), (6) and the fact that h(i) is non-decreasing and $v_{n,\alpha}(i,r)$ is non-negative.

2. Comparison of Socially Optimal and Individually Optimal Joining Policies

In this section we characterize the joining policy of an individual customer who acts in his own interest, and compare it to a socially optimal policy. It will be seen that an individually optimal policy is less selective (admits more customers) for each i, n, and α, than a socially optimal policy.

Let $C_{n,\alpha}(i)$ be the expected α-discounted holding cost incurred by a customer who joins the system when i customers are already present and n periods remain in the horizon. Then an individually optimal joining policy for a customer with reward r, arriving when i customers are in the system and n periods remain, takes the form:

enter iff $r \geq C_{n,\alpha}(i)$.

Throughout this section we shall make the following simplifying assumptions:

(i) the queue discipline is first-in, first-out (FIFO);

(ii) the holding cost of each customer is proportional to the length of time he spends in the system, with the same constant of proportionality, h, for all customers;

(iii) a customer who is present at the end of the horizon remains in the system until he is served.

Assumption (i) implies that a customer's holding time is not affected by the behavior of future arrivals, and hence his decision whether or not to enter need not be based on any presumptions or guesses about such behavior. Assumption (ii) implies that the holding cost per unit time to society as a whole while i customers are present is proportional to i; that is, $h(i) = h \cdot i$. Without loss of generality, we assume henceforth that $h = 1$.

Assumptions (i), (ii), and (iii) together imply that the holding cost of a customer who enters when i are in the system is in fact independent of the horizon length n. Specifically, let $\{S_k, k \geq 1\}$ be a sequence of i.i.d. random variables, each exponentially distributed with mean $1/\mu$, and set $s_i = S_1 + S_2 + \ldots + S_i$, $i \geq 1$.

Then $C_{n,\alpha}(i) = C_\alpha(i)$, for all $n \geq 0$, where

$$(12) \qquad C_\alpha(i) = E\{\int_0^{S_{i+1}} e^{-\alpha t}\, dt\}$$
$$= \frac{1}{\alpha + \mu} \sum_{j=o}^{i} \left(\frac{\mu}{\alpha + \mu}\right)^j$$
$$= [1 - \left(\frac{\mu}{\alpha + \mu}\right)^{i+1}]/\alpha$$

(cf. Theorem 7 of Lippman and Stidham [9]).

The following properties are immediate consequences of (12).

<u>Theorem 7.</u> $C_\alpha(i)$ is increasing and concave in $i \geq 0$, for each $\alpha \geq 0$, and decreasing and convex in α for each $i \geq 0$.

Thus, as expected, an individually optimal policy admits fewer customers as either the number in the system increases or the interest rate decreases.

The comparison between socially optimal and individually optimal policies is made easier if we first eliminate from $V_{n,\alpha}(i,r)$ the (fixed) holding costs due to the i customers already in the system. The following development parallels that in Lippman and Stidham [9] quite closely.

Let $h_{n,\alpha}(i)$ be the expected α-discounted holding cost incurred if there are i customers in the system, n periods remain in the horizon, and costs of future customers are ignored. By assumption (i), $h_{n,\alpha}(i)$ is a fixed cost, which is not affected by the admission decisions over the remaining horizon. By assumption (iii), this cost includes the holding cost incurred after the end of the horizon by any of the i customers still in the system at that time. Thus, $h_{n,\alpha}(i) = h_\alpha(i)$, independent of n, where (by assumption (ii)) $h_\alpha(i) = E\{\int_0^\infty e^{-\alpha t}\,(i - N(t))^+\, dt\}$. Finally, assumption (i) implies that $C_\alpha(i) = h_\alpha(i + 1) - h_\alpha(i)$, since the assumption that a customer's holding time is not affected by future arrivals implies that the increment in fixed costs incurred in going from state i to i + 1 is borne entirely by the entering customer. (This relation also follows directly from the expressions for $C_\alpha(i)$ and $h_\alpha(i)$,

since $(i + 1 - N(t))^+ = (i - N(t))^+ + 1$, for $0 \leq t < s_{i+1}$, and $(i + 1 - N(t))^+ = (i - N(t))^+ = 0$, for $t \geq s_{i+1}$.)

In light of assumptions (ii) and (iii), it is appropriate to assume that the terminal reward for the social optimizer, $U_{o,\alpha}(i)$, is given by (4), with linear holding cost function, $h(i) = i$. Thus $U_{o,\alpha}(i) = -h_\alpha(i)$. As a consequence of (10), therefore, we have the important relation:

(13) $\quad h_\alpha(i) = E\{ \int_0^T e^{-\alpha t} (i - N(t))^+ \, dt\} + E\{e^{-\alpha T} h_\alpha((i - N(T))^+)\}.$

Define $\tilde{V}_{n,\alpha}(i,r) = V_{n,\alpha}(i,r) + h_\alpha(i)$, $i \geq 0$, $\tilde{U}_{n,\alpha}(i) = U_{n,\alpha}(i) + h_\alpha(i)$, $n \geq 0$. (Note that $\tilde{U}_{o,\alpha} \equiv 0$.) $\tilde{V}$ represents the maximum expected α-discounted controllable net benefit, since $h_\alpha(i)$ is an unavoidable, fixed cost. Using (13) and the relation between $h_\alpha(i)$ and $C_\alpha(i)$, we can derive the following recursive equations, which are equivalent to (1) and (2):

(14) $\quad \tilde{V}_{n,\alpha}(i,r) = \max \{r - C_\alpha(i) + \tilde{U}_{n,\alpha}(i + 1); \tilde{U}_{n,\alpha}(i)\}, \; n \geq 0$

(15) $\quad \tilde{U}_{n,\alpha}(i) = E\{e^{-\alpha T} \tilde{V}_{n-1,\alpha}((i - N(T))^+, R)\}, \; n \geq 1.$

Note that, in effect, what is done in this recursion is to charge each customer's entire expected α-discounted holding cost at the instant he enters the system, rather than spread it out over his entire stay.

The recursion also reveals that it is socially optimal for a customer to enter iff $r \geq C_\alpha(i) + \tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$, that is, $S_{n,\alpha}(i) = C_\alpha(i) + \tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$. The difference $\tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$, represents the <u>external effect</u> of the customer's decision to enter: the expected α-discounted cost that he imposes on future arrivals. Our next result shows that this external effect is non-negative, so that $S_{n,\alpha}(i) \geq C_\alpha(i)$: a socially optimal policy admits fewer customers than an individually optimal policy.

<u>Theorem 8.</u> Given $\alpha \geq 0$, $n \geq 0$, $i \geq 0$, $\tilde{U}_{n,\alpha}(i) \geq \tilde{U}_{n,\alpha}(i + 1)$, and hence $S_{n,\alpha}(i) \geq C_\alpha(i)$.

<u>**Proof.**</u> Clearly, $\tilde{U}_{o,\alpha}(i) \geq \tilde{U}_{o,\alpha}(i + 1)$. For each $n \geq 1$, if $\tilde{V}_{n-1,\alpha}(i,r) \geq \tilde{V}_{n-1,\alpha}(i + 1, r)$, for all i, r, then (15) implies that $\tilde{U}_{n,\alpha}(i) \geq \tilde{U}_{n,\alpha}(i + 1)$. Therefore, it suffices to show that $\tilde{U}_{n,\alpha}(i) \geq \tilde{U}_{n,\alpha}(i + 1)$ implies that $\tilde{V}_{n,\alpha}(i, r) \geq \tilde{V}_{n,\alpha}(i + 1, r)$. But this is a direct consequence of (14) and the fact that $C_\alpha(i)$ is increasing in i.

3. Optimal Congestion Tolls

The socially optimal policy may be implemented by charging each entering customer a toll, or entrance fee, and then allowing him to act in his own interest. In this way, the individually optimal and socially optimal policies can be made to coincide. Our analysis thus far indicates that the optimal congestion toll is $T_{n,\alpha}(i) = S_{n,\alpha}(i) - C_{n,\alpha}(i)$, and that $T_{n,\alpha}(i) \geq 0$. The toll is exactly equal to the external effect, $\tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$, caused by the customer's decision to enter.

Lippman and Stidham [9] examined optimal congestion tolls in the context of exponential congestion systems, in which the system is observed at the epochs of departures and certain null events as well as arrivals, and there are no terminal costs, that is, customers present at the end of the horizon depart immediately rather than remain in the system until their services are completed. In that context it was found that, contrary to intuition, the optimal congestion toll is not monotonic in i. This same counter-intuitive result holds in the system under study in this paper, but, as we shall see, for rather different reasons.

To examine further the properties of optimal congestion tolls, first observe that, since $S_{n,\alpha}(i) = C_{\alpha}(i) + \tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$, (14) is equivalent to

$$\tilde{V}_{n,\alpha}(i,r) = (r - S_{n,\alpha}(i))^{+} + \tilde{U}_{n,\alpha}(i),$$

where $x^{+} = \max(x,0)$. This relation can be combined with (15) to yield ($n \geq 1$)

$$\tilde{U}_{n,\alpha}(i) = E\{e^{-\alpha T}[R - S_{n-1,\alpha}((i - N(T))^{+})]^{+}\} + E\{e^{-\alpha T}\,\tilde{U}_{n-1,\alpha}((i - N(T))^{+}\}.$$

Since $T_{n,\alpha}(i) = \tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i + 1)$, we conclude that, for $n \geq 1$, $i \geq 0$, ($T_{o,\alpha} \equiv 0$)

$$(16)\quad T_{n,\alpha}(i) = E\{e^{-\alpha T}g_{n-1,\alpha}(i,T)\} + E\{e^{-\alpha T}T_{n-1,\alpha}(i - N(T));\ N(T) \leq i\},$$

where $g_{n-1,\alpha}(i,t) = E\{[R-S_{n-1,\alpha}((i - N(t))^+)]^+\} - E\{[R-S_{n-1,\alpha}((i+1 - N(t))^+)]^+\}$

$$= E\{\int_{S_{n-1}((i - N(t))^+)}^{S_{n-1}((i+1 - N(t))^+)} Pr\{R > r\}\, dr\}.$$

Using (16), (12), and the relation $S_{n,\alpha}(i) = C_\alpha(i) + T_{n,\alpha}(i)$, one can calculate recursively the values of the optimal congestion tolls for $n = 1,2,\ldots$, starting with $T_{o,\alpha}(i) = 0$. Note that to calculate $T_{n,\alpha}(i)$ one needs the values of $T_{n-1,\alpha}(j)$, for $0 \leq j \leq i+1$, and in general, $T_{n-k,\alpha}(j)$, for $0 \leq j \leq i+k$ ($k = 1, 2, \ldots, n$).

In the single-server version of the exponential congestion system studied by Lippman and Stidham [9], it was discovered that $T_{n,\alpha}(i) = 0$, for $i+1 \geq n-1$, and consequently $T_{n,\alpha}(i)$ cannot be monotonic in i. The reason for this somewhat unsuspected phenomenon is that in this case the holding times of future customers are not affected by whether the customer who has just arrived enters or not. Each of them, if he enters, must remain in the system at least until the end of the horizon, because at most one service completion can occur in each period. Moreover, no service completions are allowed after the end of the horizon.

In the system currently under study, both these properties are absent. Because the system is observed only at arrival points and service completions are governed by a Poisson process, arbitrarily many services can be completed in a period. Even if this were not the case, the decision of an arriving customer to enter or not would still affect future customer's holding times, since customers already in the system remain in the system until they are served, no matter how few periods remain. For both these reasons, then, we should expect $T_{n,\alpha}(i)$ to remain positive, no matter how large i is.

On the other hand, it is clear that, for any fixed n, $T_{n,\alpha}(i)$ approaches zero asymptotically as i approaches infinity. To see this, recall that $T_{n,\alpha}(i) = \tilde{U}_{n,\alpha}(i) - \tilde{U}_{n,\alpha}(i+1)$. Now, it is clear from their definitions that both $\tilde{V}_{n,\alpha}(i,r)$ and $\tilde{U}_{n,\alpha}(i)$ are non-negative. But $\tilde{U}_{n,\alpha}(i)$ is non-increasing in i.

Therefore, $\tilde{U}_{n,\alpha}(i)$ approaches a finite limit, and hence $T_{n,\alpha}(i)$ approaches zero, as i approaches infinity. We conclude that, just as in the system studied by Lippman and Stidham [9], it is impossible for $T_{n,\alpha}(i)$ to be monotonically increasing in i. Nor is $T_{n,\alpha}(i)$ generally decreasing in i. To see this, suppose $n = 1$, in which case

$$T_{1,\alpha}(i) = E\{e^{-\alpha T} g_{o,\alpha}((i - N(T))^{+})\}.$$

Since $T_{o,\alpha}(i) = 0$, $S_{o,\alpha}(i) = C_{\alpha}(i)$, for all $i \geq 0$. Now assume that $Pr\{R > r\} = 1$ for $r \geq C_{\alpha}(i+1)$. Then $g_{o,\alpha}(i,t) = C_{\alpha}((i+1 - N(t))^{+}) - C_{\alpha}((i - N(t))^{+})$ and

$$\begin{aligned}
T_{1,\alpha}(i) &= E\{e^{-\alpha T}[C_{\alpha}((i + 1 - N(T))^{+}) - C_{\alpha}((i - N(T))^{+})]\} \\
&= E\{e^{-\alpha T}[C_{\alpha}(i + 1 - N(T)) - C_{\alpha}(i - N(T))];\ N(T) < i\} \\
&= E\{e^{-\alpha T} \sum_{k=0}^{i} [C_{\alpha}(i + 1 - k) - C_{\alpha}(i - k)]\, e^{-\mu T}(\mu T)^{k}/k!\} \\
&= E\{e^{-\alpha T} \sum_{k=0}^{i} \frac{1}{\alpha + \mu} \left(\frac{\mu}{\alpha + \mu}\right)^{i+1-k} e^{-\mu T}(\mu T)^{k}/k!\} \\
&= \frac{1}{\alpha + \mu} \left(\frac{\mu}{\alpha + \mu}\right)^{i+1} E\{ \sum_{k=0}^{i} e^{-(\alpha+\mu)T} ((\alpha + \mu)T)^{k}/k!\}
\end{aligned}$$

Now suppose T is exponentially distributed with parameter λ. Then

$$T_{1,\alpha}(i) = \frac{1}{\alpha + \mu} \left(\frac{\lambda}{\alpha + \lambda + \mu}\right) \left(\frac{\mu}{\alpha + \mu}\right)^{i+1} \sum_{k=0}^{i} \left(\frac{\alpha + \mu}{\alpha + \lambda + \mu}\right)^{k}.$$

Therefore

$$T_{1,\alpha}(i) - T_{1,\alpha}(0) = \frac{1}{\alpha + \mu} \left(\frac{\lambda}{\alpha + \lambda + \mu}\right) \left(\frac{\mu}{\alpha + \mu}\right) \left[\frac{\mu}{\alpha \cdot \mu} + \frac{\mu}{\alpha + \lambda + \mu} - 1\right].$$

For any fixed values of λ and μ, the term in brackets is positive for small α and negative for large α. Hence, $T_{1,\alpha}(i)$ is not monotonic in i.

4. Optimal Policies for the Infinite Horizon

In this section we study socially optimal policies for infinite-horizon problems, both with and without discounting. In contrast to Section 2, we do not make assumptions (i), (ii), and (iii), but rather return to the generality of Section 1. In particular, the holding cost rate $h(i)$ is once again allowed to be a general convex, non-decreasing function of i, the number of customers present.

We first consider the infinite-horizon problem with discounting. We assume that $\beta = E\{e^{-\alpha T}\} < 1$. A policy π is defined in the usual way (cf. [11],[6], [8], [12]) as a sequence of decision rules, possibly randomized and possibly dependent on the history of the process as well as the current state, (i,r), for selecting actions ($a = 0$ or 1) at each decision point. A stationary policy is a deterministic policy that always takes the same action, $a = f(i,r)$, whenever the state is (i,r). Let $V_\alpha^\pi(i,r)$ be the expected α-discounted net benefit over an infinite horizon from following policy π, when the starting state is (i,r). That is, $V_\alpha^\pi(i,r)$ is the expected sum of the discounted net returns (rewards minus holding costs) earned in all periods $k = 1, 2, \ldots$.

To see that $V_\alpha^\pi(i,r)$ is well defined, first note that the positive part of the discounted net return in period k is bounded above by $e^{-\alpha t_k} R_k^+$, where t_k and R_k are the arrival times and rewards, respectively, of customer k. Since t_k and R_k do not depend on the policy in effect, this bound holds for all policies. Consequently, the expected discounted positive part of the net return in period k, under any policy, is bounded above by $\beta^k E\{R^+\}$, where $\beta < 1$ and $E\{R^+\} < \infty$, by assumption. It follows that condition (3) of Serfozo [12] is satisfied, so that Serfozo's Assumptions (A1) and (A2) hold. Hence, $V_\alpha^\pi(i,r)$ is well defined. (It may equal $-\infty$.)

Define $V_\alpha(i,r) = \sup_\pi V_\alpha^\pi(i,r)$. $V_\alpha(i,r)$ is the optimal return function for the infinite-horizon problem with discounting. An application of Theorems 2.1 and 2.2 of Serfozo [12] then yields:

Theorem 9. $V_\alpha(i,r)$ **satisfies the functional equation**

(17) $$V_\alpha(i,r) = \max\{r + U_\alpha(i+1),\ U_\alpha(i)\}$$

where

(18) $$U_\alpha(i) = -E\{\int_0^T e^{-\alpha t} h((i - N(t))^+)dt\} + E\{e^{-\alpha T} V_\alpha((i-N(T))^+)\}.$$

Moreover, the stationary policy that accepts a customer iff $r \geq S_\alpha(i) \equiv U_\alpha(i) - U_\alpha(i+1)$ is optimal among all policies.

The next theorem shows that the monotonicity of optimal policies carries over from finite to infinite horizon.

Theorem 10. $V_\alpha(i,r)$ **and** $U_\alpha(i)$ **are concave in i, so that** $S_\alpha(i)$ **is non-decreasing in i and hence a critical-number policy is optimal.**

Proof. Consider a sequence of finite-horizon problems, in which the terminal reward, $U_{o,\alpha}(i)$, is identically zero. It follows from Theorem 2.2 (c) of Serfozo [12] that $V_\alpha(i,r) = \lim_{n\to\infty} V_{n,\alpha}(i,r)$ and hence $U_\alpha(i) = \lim_{n\to\infty} U_{n,\alpha}(i)$. Concavity of $V_\alpha(i,r)$ and $U_\alpha(i)$ then follows from Theorem 3.

The above proof shows that the optimal return function, $V_\alpha(i,r)$, and the optimal reward-cutoff point, $S_\alpha(i)$, can be approximated by solving a sequence of finite-horizon problems, with $U_{o,\alpha}(i) \equiv 0$. The question remains as to whether the method of successive approximations works for other terminal reward functions. (This is a non-trivial problem, since the one-period net return is unbounded.) The following theorem gives a partial answer to this question.

Theorem 11. Suppose that condition (4) holds; that is, $U_{o,\alpha}(i)$ is minus the expected α-discounted holding cost incurred until the i customers still in the system are served. Suppose also that $\beta^n h(n) \to o$ as $n \to \infty$. Then $V_{n,\alpha}(i,r) \uparrow V_\alpha(i,r)$, $U_{n,\alpha}(i) \uparrow U_\alpha(i)$, and $S_{n,\alpha}(i) \uparrow S_\alpha(i)$, as $n \to \infty$.

Proof. For each (i,r) and $n \geq 0$, $U_{n,\alpha}(i) \leq \beta E\{R^+\}/(1-\beta) < \infty$ and $V_{n,\alpha}(i,r) \leq r + \beta E\{R^+\}/(1-\beta) < \infty$. Moreover, $U_{n,\alpha}(i)$ and $V_{n,\alpha}(i,r)$ are both non-decreasing in n, as may be shown by a simple induction argument using (1) and (2). (To get started, one must show that $U_{1,\alpha}(i) \geq U_{o,\alpha}(i)$. But $V_{o,\alpha}(i,r) \geq U_{o,\alpha}(i)$ implies that $U_{1,\alpha}(i) \geq - E\{\int_0^T e^{-\alpha t} h((i - N(t))^+)dt\} + E\{e^{-\alpha T} U_{o,\alpha}((i - N(T))^+)\}$, which equals $U_{o,\alpha}(i)$ by (10).) Hence both $V_{n,\alpha}(i,r)$ and $U_{n,\alpha}(i)$ monotonically approach finite limits as $n \to \infty$. Moreover, it follows from (1) and (2) that these limits, say $V'_\alpha(i,r)$ and $U'_\alpha(i)$, satisfy (17) and (18). If we knew that the solutions to (17) and (18) were unique, then Theorem 9 would imply that $V'_\alpha(i,r) = V_\alpha(i,r)$ and $U'_\alpha(i) = U_\alpha(i)$, and we would be done. However, $V_\alpha(i,r)$ and $U_\alpha(i)$ need not be the unique solutions to (17) and (18), since the one-period net return is not bounded. One could establish uniqueness by imposing further conditions, such as the existence of a polynomial bound on the one-period net return, and then applying the results of Lippman [8]. Instead of this approach, we shall employ a more direct argument to show that $V'_\alpha(i,r) = V_\alpha(i,r)$ (and hence $U'_\alpha(i) = U_\alpha(i)$).

To this end, first observe that, under the assumption that $U_{o,\alpha}(i)$ is given by (4), $V_{n,\alpha}(i,r)$ is the expected α-discounted total net benefit over an infinite horizon from following a particular policy, namely the policy π_n which takes the optimal actions for the n-period problem at the first n decision points, and then admits no further customers. Thus, $V_\alpha(i,r) \geq V_{n,\alpha}(i,r)$. Now let $V^\circ_{n,\alpha}(i,r)$ be the optimal expected α-discounted total net-benefit over n periods when the terminal reward is identically zero. (We know from the proof of Theorem 10 that $V^\circ_{n,\alpha}(i,r) \to V_\alpha(i,r)$ and $U^\circ_{n,\alpha}(i) \to U_\alpha(i)$.) Let π°_n be the policy that achieves $V^\circ_{n,\alpha}(i,r)$, that is, the policy that is optimal for the n-period problem with terminal reward identically zero. Now consider what happens if we employ policy π°_n for the n-period problem when the terminal reward is $U_{o,\alpha}(i)$, as given by (4). The expected α-discounted total net benefit over n periods from following π°_n is in this case

$V^{\circ}_{n,\alpha}(i,r) + E_{\pi^{\circ}_n}\{e^{-\alpha t_n} U_{o,\alpha}(X_n(i,r))\}$, where $X_n(i,r)$ = number of customers present at the end of the n-period horizon, given the initial state is (i,r), and $E_{\pi^{\circ}_n}\{\cdot\}$ denotes the expected value under policy π°_n. But, since π_n is optimal for the n-period problem with terminal reward $U_{o,\alpha}(i)$, we conclude that

$$(19) \quad V^{\circ}_{n,\alpha}(i,r) + E_{\pi^{\circ}_n}\{e^{-\alpha t_n} U_{o,\alpha}(X_n(i,r))\} \leq V_{n,\alpha}(i,r) \leq V_{\alpha}(i,r).$$

Now from the definition of $U_{o,\alpha}(i)$,

$$\begin{aligned} 0 &\geq E_{\pi^{\circ}_n}\{e^{-\alpha t_n} U_{o,\alpha}(X_n(i,r))\} \\ &\geq E\{e^{-\alpha t_n} U_{o,\alpha}(i+n+1)\} \\ &= \beta^n U_{o,\alpha}(i+n+1) \\ &= -\beta^n E\{\int_0^{\infty} e^{-\alpha t} h((i+n+1-N(t))^+)dt\} \\ &\geq -\beta^n h(i+n+1)/\alpha. \end{aligned}$$

It follows, therefore, from (10) and the assumption that $\beta^n h(n) \to 0$ that $V_{\alpha}(i,r) = \lim_{n\to\infty} V^{\circ}_{n,\alpha}(i,r) \leq \lim_{n\to\infty} V_{n,\alpha}(i,r) \leq V_{\alpha}(i,r)$, so that $\lim_{n\to\infty} V_{n,\alpha}(i,r) = V_{\alpha}(i,r)$, the desired result.

We now turn our attention to the infinite-horizon undiscounted problem ($\alpha = 0$) in which the objective is to maximize long-run average expected return per unit time. For the remainder of the paper we assume that the holding cost is linear : $h(i) \equiv i$.

<u>Theorem 12.</u> Suppose the holding cost is linear: $h(i) \equiv i$. Then the functions $V(i,r) \equiv \lim_{\alpha\to 0+} (V_{\alpha}(i,r) - U_{\alpha}(0))$, $U(i) \equiv \lim_{\alpha\to 0+} (U_{\alpha}(i) - U_{\alpha}(0))$ are well defined, concave, and satisfy the functional equation

$$(20) \quad \begin{aligned} V(i,r) &= \max\{r + U(i+1), U(i)\} \\ U(i) &= -E\{\int_0^T (i - N(t))^+ dt\} + E\{V((i-N(T))^+, R\} - g E\{T\}, \end{aligned}$$

where g is the maximal long-run average expected return per unit time. Moreover, the stationary policy associated with $S(i) \equiv \lim_{\alpha \to 0+} S_\alpha(i) = U(i+1) - U(i)$ is average-return optimal and attains the maximum in (20).

Proof. The proof uses results of Lippman [6], [8] and parallels that of Theorem 7 of Lippman and Stidham [9]. First observe that, for each $\alpha > 0$, $S_\alpha(i) \geq C_\alpha(i) = (1 - (\mu/(\alpha+\mu))^{i+1})/\alpha$ and that $C_\alpha(i) \uparrow (i+1)/\mu$, as $\alpha \to 0+$. Now choose M so large that $\lambda(1-F(M)) < \mu$ and choose $\hat{i}$ so large that $(\hat{i}+1)/\mu > M$. Then it is possible to find an $\alpha_0 > 0$ such that $\alpha \leq \alpha_0$ implies that $C_\alpha(\hat{i}) \geq M$, and hence $C_\alpha(i) \geq C_\alpha(\hat{i}) \geq M$ for all $i \geq \hat{i}$. Combining these results, we conclude that there exists an $\alpha_0 > 0$ and $\hat{i} < \infty$ such that

$$(21) \qquad \lambda(1 - F(S_\alpha(i))) \leq \lambda(1 - F(S_\alpha(\hat{i}))) < \mu, \text{ for all } \alpha \leq \alpha_0,\ i \geq \hat{i}.$$

Now let π_n, $n \geq 1$, be the stationary policy associated with $S_{\alpha_0/n}$, so that π_n is optimal for the infinite-horizon problem with discount rate $\alpha = \alpha_0/n$. Let $P(\pi_n)$ be the transition probability matrix for the Markov chain, imbedded at arrival points, associated with π_n. It follows from (21) that $\{P(\pi_n), n \geq 0\}$ is uniformly positive recurrent, so that starting in state i, both the expected time and the expected value of the sum of customer rewards and the absolute value of customer holding costs incurred until the first visit to state 0 are uniformly bounded in n. The remainder of the proof is identical to that of Theorem 7 of [9].

Optimal Congestion Tolls for the Infinite Horizon

The relationship between socially optimal and individually optimal joining policies, discussed in Section 2 for finite-horizon problems, carries over in most respects to infinite-horizon problems.

To see this, assume as in Section 2 that (i), (ii), and (iii) hold. First consider the problem with discounting ($\alpha > 0$) and an infinite horizon. The individually optimal policy is the same as in the case of a finite horizon, because of assumption (iii):

$$\text{enter iff } r \geq C_\alpha(i),$$

where $C_\alpha(i)$ is the expected α-discounted holding cost of a customer who enters when i customers are present, and is given by (12). Define $T_\alpha(i)$, the optimal congestion toll for the infinite-horizon problem with discounting, by

$$T_\alpha(i) \equiv S_\alpha(i) - C_\alpha(i).$$

Since $S_{n,\alpha}(i) \geq C\,(i)$ (Theorem 8) and $S_{n,\alpha}(i) \uparrow S_\alpha(i)$ as $n \to \infty$ (Theorem 11), we conclude that $T_{n,\alpha}(i) \uparrow T_\alpha(i)$ as $n \to \infty$ and $T_\alpha(i) \geq 0$, so that once again a socially optimal policy admits fewer customers than an individually optimal policy. Define $\tilde{U}_\alpha(i) \equiv U_\alpha(i) + h_\alpha(i)$. Then $\tilde{U}_\alpha(i) = \lim_{n\to\infty} \tilde{U}_{n,\alpha}(i)$ (Theorem 11) and hence $\tilde{U}_\alpha(i) \geq 0$. Moreover, $T_\alpha(i) = \tilde{U}_\alpha(i) - \tilde{U}_\alpha(i+1)$, so that $\tilde{U}_\alpha(i) \geq \tilde{U}_\alpha(i+1)$. Therefore, $\tilde{U}_\alpha(i)$ approaches a finite limit as $i \to \infty$ and hence $T_\alpha(i) \to 0$ as $i \to \infty$, so that the toll cannot be monotonically increasing, just as in the case of a finite horizon. Finally, it follows from (16) and the fact that $T_{n,\alpha}(i) \to T_\alpha(i)$ as $n \to \infty$ that $T_\alpha(i)$ satisfies the functional equation:

$$T_\alpha(i) = E\left\{e^{-\alpha T} \int_{S_a((i-N(T))^+)}^{S_a((i+1-N(T))^+)} \Pr\{R>r\}dr\right\} +$$

$$E\,\{e^{-\alpha T}\, T_\alpha(i-N(T));\; N(T) \leq i\}\,.$$

Now consider what happens in the infinite-horizon as $\alpha \to o+$ and the criterion becomes maximization of long-run average expected return for unit time. The individually optimal policy is:

$$\text{enter iff} \quad r \geq C(i),$$

where $C(i) = \lim_{\alpha \to o+} C_\alpha(i) = (i+1)/\mu$, the expected holding cost of a customer who enters when i customers are present. Define T(i), the optimal congestion toll for the infinite-horizon, average-return problem, by

$$T(i) \equiv S(i) - C(i).$$

Since $S_\alpha(i) \to S(i)$ (Theorem 12) and $C_\alpha(i) \to C(i)$ as $\alpha \to o+$, we conclude that $T(i) = \lim_{\alpha \to o+} T_\alpha(i)$ and hence $T(i) \geq 0$. It follows from (22) that T(i) satisfies the functional equation

$$\text{(23)} \quad \begin{aligned} T(i) = {} & E\left\{\int_{S((i-N(T))^+)}^{S((i+1-N(T))^+)} \Pr\{R>r\}dr\right\} + \\ & E\,\{T(i-N(T));\ N(T) \leq i\}\ . \end{aligned}$$

If we define $g(i) \equiv E\left\{\int_{S((i-N(T))^+)}^{S((i+1-N(T))^+)} \Pr\{R>r\}dr\right\}$ and $b(k) \equiv \Pr\{N(T)=k\}$, $k \geq 0$, then (23) can be rewritten as

$$\text{(24)} \quad T(i) = g(i) + \sum_{k=0}^{i} T(i-k)\, b(k),$$

which is a discrete renewal equation (Feller [3], p. 290).

It follows from the discrete form of the Key Renewal Theorem (Feller [3], p. 291, Theorem 1) that

$$\text{(25)} \quad \lim_{i \to \infty} T(i) = \rho\, E\{(R-S(0))^+\}\ ,$$

where $\rho^{-1} = \mu\, ET = E\{N(T)\} = \sum_{k=0}^{\infty} k\, b(k)$, and

$\infty > E\{(R-S(0))^+\} = E\{\int_{S(0)}^{\infty} \Pr\{R>r\}dr\} = \sum_{i=0}^{\infty} g(i)$, by Fubini's Theorem.

Now $\rho > 0$ by assumption (since $ET < \infty$) and the case in which $E\{(R-S(0))^+\} = 0$ is of little practical interest. (In that case $g(i) \equiv 0$ and (24) implies that $T(i) \equiv 0$). Hence we conclude that in general $T(i)$ has a <u>positive</u> limit as $i \to \infty$, in contrast to the discounted and/or finite-horizon cases. It is thus theoretically possible for $T(i)$ to be monotonically increasing. One might hope to determine whether or not $T(i)$ is monotonically increasing in general by appealing to the functional equation (23), but we have been unable to do so for the case of a general interarrival-time distribution. (See below, however, for an analysis of the M/M/1 case). Incidentally, while in theory it should be possible to use (23) or (24) (or a finite truncation thereof) to solve explicitly (or approximately) for the $T(i)$'s, this approach is complicated considerably by the fact that the equations are <u>not</u> linear, since the $g(i)$'s depend on the $T(i)$'s through $S((i-N(T))^+)$ and $S((i+1-N(T))^+)$. We can answer the question of monotonicity of the tolls, and also give algorithms for their computation, if we specialize to the case of an exponential interarrival-time distribution with parameter $\lambda = (ET)^{-1}$: an M/M/1 system. In this case the system has the Markov property at epochs of service-completions and "null events" (see Lippman [7] and Lippman and Stidham [9]) as well as arrivals and it can easily be shown that the following functional equations are equivalent to (20) for the average-return problem:

$$V(i,r) = \max\{r+U(i+1),\ U(i)\}$$

$$U(i) = -\frac{i}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} E\{V(i,R)\} + \frac{\mu}{\lambda+\mu} U(i-1) - \frac{g}{\lambda+\mu},\ i \geq 1,$$

$$U(0) = \frac{\lambda}{\lambda+\mu} E\{V(0,R)\} + \frac{\mu}{\lambda+\mu} U(0) - \frac{g}{\lambda+\mu}.$$

Moreover, the functional equation (23) for T(i) is equivalent to

$$(27) \quad T(i) = \rho \int_{S(i)}^{S(i+1)} \Pr\{R>r\}dr + T(i-1)$$

(with $T(-1) = 0$). We can conclude immediately from (27) that T(i) is monotonically increasing in i in this case, since $S(i+1) \geq S(i)$.

This result, together with the fact that the toll can never be monotonically increasing when the discount rate, α, is positive, leads us to conclude that there is no strongly (Blackwell) optimal policy for this system. For, if there were, it would simultaneously be optimal for all sufficiently small $\alpha > 0$ and also optimal for the average-return problem. But this is impossible, since the characteristics of these two classes of policies are mutually inconsistent.

References

1. Adler, I., and P. Naor, "Social Optimization versus Self-Optimization in Waiting Lines," Technical Report No. 4, Department of Operations Research, Stanford University, 1969.

2. Cramer, M., "Optimal Customer Selection in Exponential Queues," ORC 71-24, Operations Research Center, University of California, Berkeley, 1971.

3. Feller, W., An Introduction to Probability Theory and its Applications, I, Wiley, New York, 1974, 3rd ed.

4. Knudsen, N.C., "Individual and Social Optimization in a Multiserver Queue with a General Cost-Benefit Structure," Econometrica, 40, 1972, 515-528.

5. Knudsen, N. C., and Stidham, S., "Individual and Social Optimization in a Birth-Death Congestion System with General Cost-Benefit Structure," Technical Report No. 76-8, Department of Industrial Engineering, North Carolina State University, August, 1976.

6. Lippman, S.A., "Semi-Markov Decision Processes with Unbounded Rewards," Management Science, 19, 1973, 717-731.

7. Lippman, S.A., "Applying a new Device in the Optimization of Exponential Queueing Systems," Opns. Res., 23, 1975, 687-710.

8. Lippman, S.A., "On Dynamic Programming with Unbounded Rewards," Mgt. Sci., 21, 1975, 1225-1233.

9. Lippman, S.A., and Stidham, S., "Individual versus Social Optimization in Exponential Congestion Systems," Technical Report No. 230, Department of Operations Research, Cornell University, April, 1975 (to appear in Opns. Res.).

10. Naor, P., "On the Regulation of Queue Size by Levying Tolls," Econometrica, 37, 1969, 15-24.

11. Ross, S., Applied Probability with Optimization Applications, Holden-Day, San Francisco, 1970.

12. Serfozo, R.F., "Monotone Optimal Policies for Markov Decision Processes," Department of Industrial Engineering and Operations Research, Syracuse University, February, 1976.

13. Yechiali, U., "On Optimal Balking Rules and Toll Charges in a GI/M/1 Queueing Process," Operations Research, 19, 1971, 349-370.

14. Yechiali, U., "Customers' Optimal Joining Rules for the GI/M/s Queue," Management Science, 18, 1972, 434-443.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #74 14 TR-74 / 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 THE GI/M/1 QUEUE WITH CONTROLLED ARRIVALS.	5. TYPE OF REPORT & PERIOD COVERED 9 TECHNICAL REP[redacted]T,
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 SHALER STIDHAM, JR	8. CONTRACT OR GRANT NUMBER(s) 15 N00014-76-C-0418
9. PERFORMING ORGANIZATION NAME AND ADDRESS DEPT. OF OPERATIONS RESEARCH STANFORD UNIVERSITY STANFORD, CALIFORNIA	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-047-061
11. CONTROLLING OFFICE NAME AND ADDRESS Logistics & Mathematical Statistics Branch Code 434 Office of Naval Research Arlington, Virginia 22217	12. REPORT DATE 11 21 Jul[redacted]77
	13. NUMBER OF PAGES 27 12 32p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, Distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This research was supported in part by National Science Foundation Grant ENG 75-14847 Department of Operations Research, Stanford University and issued as Technical report #44.
Also, supported in part by a grant from the Bank of America Foundation at Stanford University

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

QUEUEING THEORY	OPTIMAL CONTROL OF QUEUEING SYSTEMS
CONTROL OF QUEUES	SOCIAL VERSUS INDIVIDUAL OPTIMALITY
	GI/M/1 QUEUES

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

SEE REVERSE SIDE

402766

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Abstract

A GI/M/1 queue is studied, in which the arrival process can be controlled by accepting or rejecting arriving customers or changing a toll. An entering customer receives a (random) reward and there is a holding cost, which is convex in the number of customers present. ~~We compare~~ socially and individually optimal joining policies are compared for finite and infinite horizons, with and without discounting. ~~We show that~~ a socially optimal policy is shown to be less likely to accept a customer than an individually optimal policy, and both policies are less likely to accept as the number of customers present increases, the horizon length increases, or the discount rate decreases. The properties of optimal congestion tolls are examined and it is discovered that the toll is not monotonic when either the horizon is finite or the discount rate is positive, whereas it can be monotonic for infinite-horizon, average-return problems.

#74/44